Acústica 2008
20 - 22 de Outubro , Coimbra, Portugal
Universidade de Coimbra

ON THE STABILITY OF THE MOTION FOR A NEO-HOOKEAN SYSTEM

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Abstract
It is well known that the vibrations are a major cause for the instability in the mechanical systems and major source of noises. In this paper we propose a simple system with two degrees of freedom based on a non-linear elastic element. For this system, it is proved in the present paper that the motion is stable, but not asymptotically stable. A comparison between the non-linear case and the linear case is performed, and for the both cases the eigenpulsations are also determined. All theoretical results are validated by numerical simulation.

Keywords: neo-Hookean, motion, stability, numerical validation.

1. Introduction

The system purposed for the study is described in figure 1, *a*. It consists of the masses m_1 and m_2 linked one to another by the linear spring of stiffness k . The mass m_1 can be considered to be the foundation of the machine-tool, and the mass m_2 the machine-tool itself. The mass m_2 is linked to the ground by the non-linear spring 1 for which the elastic force writes

$$F = k_1 x - \frac{\varepsilon_1}{x^2}, \tag{1}$$

where x is the elongation of the spring.
The fundamental working hypothesis is that

$$\varepsilon_1 > 0 . \tag{2}$$

The system has two degrees of freedom, that is the displacements z_1 and z_2 of the two masses in the vertical direction.

2. The equations of motion

Isolating the two masses (fig. 1, *b*), one obtains the differential equations of the motion

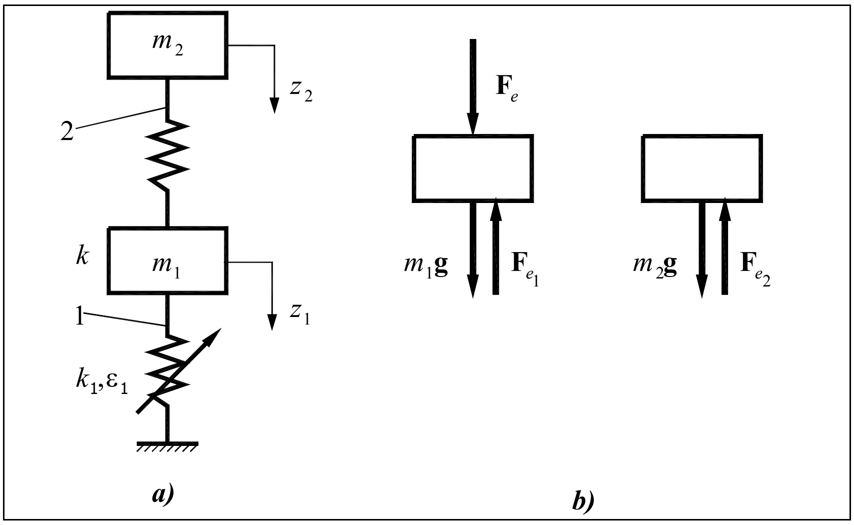

Figure 1 – The mathematical model

$$m_1\ddot{z}_1 = -k_1 z_1 + \frac{\varepsilon_1}{z_1^2} + k(z_2 - z_1) + m_1 g\,,\ m_2\ddot{z}_2 = m_2 g - k(z_2 - z_1), \tag{3}$$

where g is the gravitational acceleration.
Denoting

$$\xi_1 = z_1\,,\ \xi_2 = z_2\,,\ \xi_3 = \dot{z}_1\,,\ \xi_4 = \dot{z}_2\,, \tag{4}$$

the relations (3) transform in a system of four first order non-linear differential equations,

$$\dot{\xi}_1 = \xi_3\,,\ \dot{\xi}_2 = \xi_4\,,\ \dot{\xi}_3 = \frac{1}{m_1}\left[-k_1\xi_1 + \frac{\varepsilon_1}{\xi_1^2} + k(\xi_2 - \xi_1) + m_1 g\right],\ \dot{\xi}_4 = \frac{1}{m_2}\left[m_2 g - k(\xi_2 - \xi_1)\right]. \tag{5}$$

3. The equilibrium positions

These positions found at the intersections of the nullclines, so that one obtains the system

$$\xi_3 = 0\,,\ \xi_4 = 0\,,\ -k_1\xi_1 + \frac{\varepsilon_1}{\xi_1^2} + k(\xi_2 - \xi_1) + m_1 g = 0\,,\ m_2 g - k(\xi_2 - \xi_1) = 0\,. \tag{6}$$

Summing the last two relations (6), it results

$$-k_1\xi_1 + \frac{\varepsilon_1}{\xi_1^2} + (m_1 + m_2)g = 0\,, \tag{7}$$

wherefrom

$$k_1\xi_1^3 - (m_1 + m_2)g\xi_1^2 - \varepsilon_1 = 0\,. \tag{8}$$

In the sequence of the coefficients of powers of ξ_1 in the relation (8) there exists only one variation of sign such that applying the Descartes theorem one deduces that the equation (8) has only one positive real root. Making now $\xi_1 \mapsto -\xi_1$, one obtains the equation

$$k_1\xi_1^3 + (m_1 + m_2)g\xi_1^2 + \varepsilon_1 = 0 \tag{9}$$

for which there exists no variation of sigh in the sequence of the coefficients so that the Descartes theorem assures us that we have no negative real root for the equation (8). In conclusion, the equation (8) has exactly one positive real root, name it $\overline{\xi}_1$.

The last relation (6) becomes now a linear equation in the unknown ξ_2 and therefore it has only one solution,

$$\overline{\xi}_2 = \frac{m_2 g}{k} + \overline{\xi}_1 . \tag{10}$$

We proved in this way that the system has only one equilibrium position $\left(\overline{\xi}_1, \overline{\xi}_2, 0, 0\right)$.

Let us denote by $f(\xi_1)$ the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(\xi_1) = k_1\xi_1^3 - (m_1 + m_2)g\xi_1^2 - \varepsilon_1 \tag{11}$$

for which the derivative is

$$f'(\xi_1) = 3k_1\xi_1^2 - 2(m_1 + m_2)g\xi_1 . \tag{12}$$

The equation $f'(\xi_1) = 0$ has the solutions

$$\xi_1^{(1)} = 0 ,\ \xi_1^{(2)} = \frac{2(m_1 + m_2)g}{3k_1} , \tag{13}$$

$\xi_1^{(1)}$ being a point of maximum, and $\xi_1^{(2)}$ a point of minimum. In addition,

$$f(0) = -\varepsilon_1 < 0 . \tag{14}$$

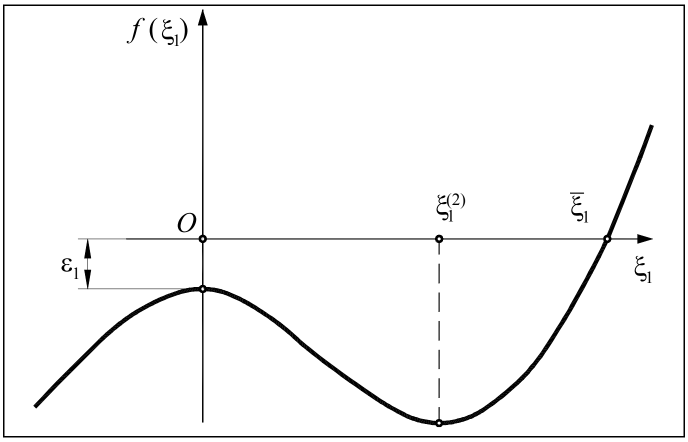

Figure 2 – The graphic of the function $f(\xi_1)$.

Graphically, the situation is presented in figure 2. It follows from here that

$$\overline{\xi}_1 > \frac{2(m_1 + m_2)g}{3k_1} . \tag{15}$$

4. The stability of the equilibrium

Let us denote by f_k, $k=\overline{1,4}$ the right-hand terms of the relations (5) and by j_{kl} the partial derivatives

$$j_{kl}=\frac{\partial f_k}{\partial \xi_l},\ k=\overline{1,4},\ l=\overline{1,4}. \tag{16}$$

We have

$$j_{11}=0,\ j_{12}=0,\ j_{13}=1,\ j_{14}=0, \tag{17}$$

$$j_{21}=0,\ j_{22}=0,\ j_{23}=0,\ j_{24}=1, \tag{18}$$

$$j_{31}=\frac{1}{m_1}\left(-k_1-k-\frac{2\varepsilon_1}{\xi_1^3}\right),\ j_{32}=\frac{k}{m_1},\ j_{33}=0,\ j_{34}=0, \tag{19}$$

$$j_{41}=\frac{k}{m_2},\ j_{42}=-\frac{k}{m_2},\ j_{43}=0,\ j_{44}=0. \tag{20}$$

The characteristic equation

$$\det(\mathbf{J}-\lambda\mathbf{I})=0 \tag{21}$$

in which $\mathbf{J}$ is the Jacobi matrix, $\mathbf{J}=\left[j_{kl}\right]_{\substack{k=\overline{1,4}\\ l=\overline{1,4}}}$, and $\mathbf{I}$ is the fourth order unity matrix, reads

$$\begin{vmatrix}-\lambda & 0 & 1 & 0\\ 0 & -\lambda & 0 & 1\\ j_{31} & j_{32} & -\lambda & 0\\ j_{41} & j_{42} & 0 & -\lambda\end{vmatrix}=0. \tag{22}$$

Multiplying the third column by λ and adding it to the first column, the fourth column by λ and adding it to the second column, results the equation

$$\lambda^4-\left(j_{31}+j_{42}\right)\lambda^2+j_{31}j_{42}-j_{32}j_{41}=0. \tag{23}$$

From the Routh–Hurwitz criterion we deduce that the equation has not all the roots with negative real part and therefore the equilibrium can not be asymptotically stable.
On the other hand, the roots of the equation (23) are

$$\lambda_{\substack{1\\2}}^2=\frac{j_{31}+j_{42}\pm\sqrt{\left(j_{31}-j_{42}\right)^2+4j_{32}j_{41}}}{2}. \tag{24}$$

Keeping into account the expressions (17)-(20), we have

$$j_{31}+j_{42}=-\frac{k_1+k+\frac{2\varepsilon_1}{\xi_1^3}}{m_1}-\frac{k}{m_2}<0, \tag{25}$$

$$4j_{32}j_{41}=\frac{4k^2}{m_1m_2}>0, \tag{26}$$

$$\left(j_{31}-j_{42}\right)^2+4j_{32}j_{41}>0, \tag{27}$$

More,

$$\left|j_{31}+j_{42}\right|>\sqrt{\left(j_{31}-j_{42}\right)^2+4j_{32}j_{41}} \tag{28}$$

because it is equivalent to

$$j_{31}j_{42}>j_{32}j_{41}, \tag{29}$$

that is

$$\frac{k_1k}{m_1m_2}+\frac{k^2}{m_1m_2}+\frac{2\varepsilon_1}{\overline{\xi}_1^3}\frac{k}{m_2}>\frac{k^2}{m_1m_2}. \tag{30}$$

The previous relations assure us that $\lambda_1^2<0$, $\lambda_2^2<0$ so that the roots of the characteristic equation (23) are all pure imaginary. The equilibrium position $\left(\overline{\xi}_1,\overline{\xi}_2,0,0\right)$ is simply stable.

5. The stability of the motion

Let $\left(\xi_1,\xi_2,\xi_3,\xi_4\right)$ a solution of the system (5) and $\left(u_1,u_2,u_3,u_4\right)$ a deviation sufficiently small in its norm. We can write

$$\begin{gathered}\dot{\xi}_1+\dot{u}_1=\xi_3+u_3,\ \dot{\xi}_2+\dot{u}_2=\xi_4+u_4,\\ \dot{\xi}_3+\dot{u}_3=\frac{1}{m_1}\left[-k_1\left(\xi_1+u_1\right)+\frac{\varepsilon_1}{\left(\xi_1+u_1\right)^2}+k\left(\xi_2-\xi_1+u_2-u_1\right)+m_1g\right],\\ \dot{\xi}_4+\dot{u}_4=\frac{1}{m_2}\left[m_2g-k\left(\xi_2-\xi_1+u_2-u_1\right)\right].\end{gathered} \tag{31}$$

Since $u_1<<\xi_1$ we can approximate

$$\frac{\varepsilon_1}{\left(\xi_1+u_1\right)^2}\approx\frac{\varepsilon_1}{\xi_1^2}-\frac{2\varepsilon_1}{\xi_1^3}u_1. \tag{32}$$

Keeping into account that $\left(\xi_1,\xi_2,\xi_3,\xi_4\right)$ is a solution of the system (5), from the relations (31) and (32) we obtain the system in deviations

$$\dot{u}_1=u_3,\ \dot{u}_2=u_4,\ \dot{u}_3=\frac{1}{m_1}\left[-k_1u_1-\frac{2\varepsilon_1u_1}{\xi_1^3}+k\left(u_2-u_1\right)\right],\ \dot{u}_4=\frac{1}{m_2}\left[-k\left(u_2-u_1\right)\right], \tag{33}$$

wherefrom

$$m_1\ddot{u}_1=-k_1u_1-\frac{2\varepsilon_1u_1}{\xi_1^3}+k\left(u_2-u_1\right),\ m_2\ddot{u}_2=-ku_2+ku_1. \tag{34}$$

From the second relation (34) we find

$$u_1 = \frac{m_2 \ddot{u}_2}{k} + u_2 ,\ \ddot{u}_1 = \frac{m_2 u_2^{(iv)}}{k} + \ddot{u}_2 , \tag{35}$$

the first relation (34) offering now

$$\frac{m_1 m_2}{k} u_2^{(iv)} + \left[\frac{m_2}{k} \left(k_1 + k + \frac{2\varepsilon_1}{\xi_1^3} \right) + m_1 \right] \ddot{u}_2 + k u_2 = 0 . \tag{36}$$

The characteristic equation reads now

$$\frac{m_1 m_2}{k} r^4 + \left[\frac{m_2}{k} \left(k_1 + k + \frac{2\varepsilon_1}{\xi_1^3} \right) + m_1 \right] r^2 + k = 0 . \tag{37}$$

The discriminate of this equation is

$$\begin{aligned} \Delta &= \left[\frac{m_2}{k} \left(k_1 + k + \frac{2\varepsilon_1}{\xi_1^3} \right) + m_1 \right]^2 - 4 m_1 m_2 \\ &= (m_2 - m_1)^2 + \left[\frac{m_2}{k} (k_1 + k) \right]^2 + 2(m_2 + m_1) \frac{m_2}{k} \left(k_1 + \frac{2\varepsilon_1}{\xi_1^3} \right) > 0 \end{aligned} \tag{38}$$

and, in addition,

$$\Delta < \left[\frac{m_2}{k} \left(k_1 + k + \frac{2\varepsilon_1}{\xi_1^3} \right) + m_1 \right]^2 . \tag{39}$$

Keeping into account that

$$a = \frac{m_2}{k} \left(k_1 + k + \frac{2\varepsilon_1}{\xi_1^3} \right) + m_1 > 0 \tag{40}$$

it immediately results that $r_1^2 < 0$, $r_2^2 < 0$ so that the roots of the characteristic equation (37) are all pure imaginary, the motion being stable, but not asymptotically stable.
The solution of the equation (36) is

$$u_2 = C_1 \cos\left(\sqrt{\frac{\frac{a - \sqrt{\Delta}}{2k}}{m_1 m_2}} t + \varphi_1 \right) + C_2 \cos\left(\sqrt{\frac{\frac{a + \sqrt{\Delta}}{2k}}{m_1 m_2}} t + \varphi_2 \right) . \tag{41}$$

By twice derivation of the expression (41), we obtain

$$\ddot{u}_2 = -\frac{\frac{a - \sqrt{\Delta}}{2k}}{m_1 m_2} C_1 \cos\left(\sqrt{\frac{\frac{a - \sqrt{\Delta}}{2k}}{m_1 m_2}} t + \varphi_1 \right) - \frac{\frac{a + \sqrt{\Delta}}{2k}}{m_1 m_2} C_2 \cos\left(\sqrt{\frac{\frac{a + \sqrt{\Delta}}{2k}}{m_1 m_2}} t + \varphi_2 \right) \tag{42}$$

and from the first relation (35) it results

$$u_1 = \left(-\frac{m_2}{k} \frac{\frac{a-\sqrt{\Delta}}{2k}}{m_1 m_2} + 1 \right) C_1 \cos\left(\sqrt{\frac{\frac{a-\sqrt{\Delta}}{2k}}{m_1 m_2}}\, t + \varphi_1 \right) + \left(-\frac{m_2}{k} \frac{\frac{a+\sqrt{\Delta}}{2k}}{m_1 m_2} + 1 \right) C_2 \cos\left(\sqrt{\frac{\frac{a+\sqrt{\Delta}}{2k}}{m_1 m_2}}\, t + \varphi_2 \right). \quad (43)$$

Everywhere C_1, C_2, φ_1 and φ_2 are constants of integration, which result from the initial conditions $u_1(0) = u_{10}$, $u_2(0) = u_{20}$, $\dot{u}_1(0) = \dot{u}_{10}$, $\dot{u}_2(0) = \dot{u}_{20}$.
The expressions (41) and (43) approximate the solution of the system in deviations (31).

6. The small oscillations around the equilibrium position

These can be obtained as a particular case of the previous paragraph for

$$\xi_1 = \bar{\xi}_1 . \quad (44)$$

Result the eigenpulsations

$$\omega_1 = \sqrt{\frac{\frac{m_2}{k}\left(k_1 + k + \frac{2\varepsilon_1}{\bar{\xi}_1^3}\right) + m_1 - \sqrt{\left[\frac{m_2}{k}\left(k_1 + k + \frac{2\varepsilon_1}{\bar{\xi}_1^3}\right) + m_1\right]^2 - 4m_1 m_2}}{\frac{2k}{m_1 m_2}}},$$
$$\omega_2 = \sqrt{\frac{\frac{m_2}{k}\left(k_1 + k + \frac{2\varepsilon_1}{\bar{\xi}_1^3}\right) + m_1 + \sqrt{\left[\frac{m_2}{k}\left(k_1 + k + \frac{2\varepsilon_1}{\bar{\xi}_1^3}\right) + m_1\right]^2 - 4m_1 m_2}}{\frac{2k}{m_1 m_2}}} \quad (45)$$

7. Comparison with the linear case

The linear case is obtained for $\varepsilon_1 = 0$.
The equation (7) writes

$$-k_1 \xi_1 + (m_1 + m_2) g = 0 , \quad (46)$$

with the solution

$$\bar{\xi}_1^{(l)} = \frac{m_1 + m_2}{k_1} g . \quad (47)$$

One observes that $\bar{\xi}_1^{(l)} < \bar{\xi}_1$ for which holds true the relation (15).
The relation (10) offers

$$\bar{\xi}_2^{(l)} = \frac{m_2 g}{k} + \frac{m_1 + m_2}{k_1} g \quad (48)$$

and therefore $\overline{\xi}_2^{(l)} < \overline{\xi}_2$, too.
The equilibrium remains again simply stable because the relations (25)-(30) still hold true.
The motion is again simply stable and we have in addition

$$\begin{aligned}\Delta^{(l)} &= \left(m_2 + m_1 + m_2 \frac{k_1}{k}\right)^2 - 4m_1 m_2 \\ &= (m_2 - m_1)^2 + \left(m_2 \frac{k_1}{k}\right)^2 + 2(m_2 + m_1) m_2 \frac{k_1}{k},\end{aligned} \tag{49}$$

$$a^{(l)} = m_2 + m_1 + \frac{m_2 k_1}{k}, \tag{50}$$

with

$$\Delta^{(l)} > 0, \quad \Delta^{(l)} < \Delta,\ a^{(l)} > 0,\ a^{(l)} < a. \tag{51}$$

The eigenpulsations are

$$\begin{aligned}\omega_1^{(l)} &= \sqrt{\frac{m_2 + m_1 + \frac{m_2 k_1}{k} - \sqrt{\left(m_2 + m_1 + \frac{m_2 k_1}{k}\right)^2 - 4m_1 m_2}}{\frac{2k}{m_1 m_2}}}, \\ \omega_2^{(l)} &= \sqrt{\frac{m_2 + m_1 + \frac{m_2 k_1}{k} + \sqrt{\left(m_2 + m_1 + \frac{m_2 k_1}{k}\right)^2 - 4m_1 m_2}}{\frac{2k}{m_1 m_2}}}.\end{aligned} \tag{52}$$

8. Numerical application

Let us consider the case

$$m_1 = 2000\,\text{kg},\ m_2 = 1000\,\text{kg},\ g = 10\,\text{m/s}^2,\ k_1 = 10^6\ \text{N/m},\ \varepsilon_1 = 700\,\text{Nm}^2,\ k = 10^5\ \text{N/m}. \tag{53}$$

The equation (8) leads us to

$$10^6 \xi_1^3 - 3000 \times 10 \xi_1^2 - 700 = 0 \tag{54}$$

with the solution

$$\xi_1 = 0.1\,\text{m}. \tag{55}$$

The relation (10) offers us

$$\xi_2 = \frac{1000 \times 10}{10^5} + 0.1 = 0.2\ \text{m}. \tag{56}$$

The expression (15) assures us that

$$0.1\,\mathrm{m} > \frac{2\times(2000+1000)\times 10}{3\times 10^6} = 0.02\,\mathrm{m}\,. \tag{57}$$

The equations (17)-(20) lead us to

$$j_{31} = \frac{1}{2000}\times\left(-10^6 - 10^5 - \frac{2\times 700}{0.1^3}\right) = -1250\,,\ j_{32} = \frac{10^5}{2000} = 50\,,\ j_{41} = \frac{10^5}{1000} = 100\,,$$
$$j_{42} = -\frac{10^5}{2000} = -50\,, \tag{58}$$

the roots of the characteristic equation (23) being (24)

$$\lambda_1^2 = \frac{-1250 - 50 \pm \sqrt{(-1250+50)^2 + 4\times 50\times 100}}{2}\,,\ \lambda_1^2 = -45.848\,,\ \lambda_2^2 = -1254.152\,. \tag{59}$$

The parameters a and Δ are

$$a = \frac{1000}{10^5}\left(10^6 + 10^5 + \frac{2\times 700}{0.1^3}\right) + 2000 = 27000\,, \tag{60}$$

$$\Delta = \left[\frac{1000}{10^5}\left(10^6 + 10^5 + \frac{2\times 700}{0.1^3}\right) + 2000\right]^2 - 4\times 2000\times 1000 = 721000000\,. \tag{61}$$

The eigenpulsations read

$$\omega_1 = \sqrt{\frac{27000 - \sqrt{721000000}}{\frac{2\times 10^5}{2000\times 1000}}} = 38.543\,\mathrm{s}^{-1}\,,\ \omega_1 = \sqrt{\frac{27000 + \sqrt{721000000}}{\frac{2\times 10^5}{2000\times 1000}}} = 733.835\,\mathrm{s}^{-1} \tag{62}$$

In the linear case we have

$$\xi_1^{(l)} = \frac{2000+1000}{10^6}\times 10 = 0.03\,\mathrm{m}\,,\ \xi_2^{(l)} = \frac{1000\times 10}{10^5} + \frac{2000+1000}{10^6}\times 10 = 0.13\,\mathrm{m}\,, \tag{63}$$

$$a^{(l)} = 1000 + 2000 + \frac{1000\times 10^6}{10^5} = 13000\,, \tag{64}$$

$$\Delta^{(l)} = \left(1000 + 2000 + 1000\times\frac{10^6}{10^5}\right)^2 - 4\times 2000\times 1000 = 161000000\,, \tag{65}$$

$$\omega_1^{(l)} = \sqrt{\frac{13000 - \sqrt{161000000}}{\frac{2\times 10^5}{2000\times 1000}}} = 55.805\,\mathrm{s}^{-1}\,,\ \omega_2^{(l)} = \sqrt{\frac{13000 + \sqrt{161000000}}{\frac{2\times 10^5}{2000\times 1000}}} = 506.839\,\mathrm{s}^{-1}\,. \tag{66}$$

One observes that

$$\omega_1 < \omega_1^{(l)}\,,\ \omega_2 > \omega_2^{(l)}\,, \tag{67}$$

so that the non-linearity has as effect the increasing of the domain of pulsations where the resonance doesn’t appear.

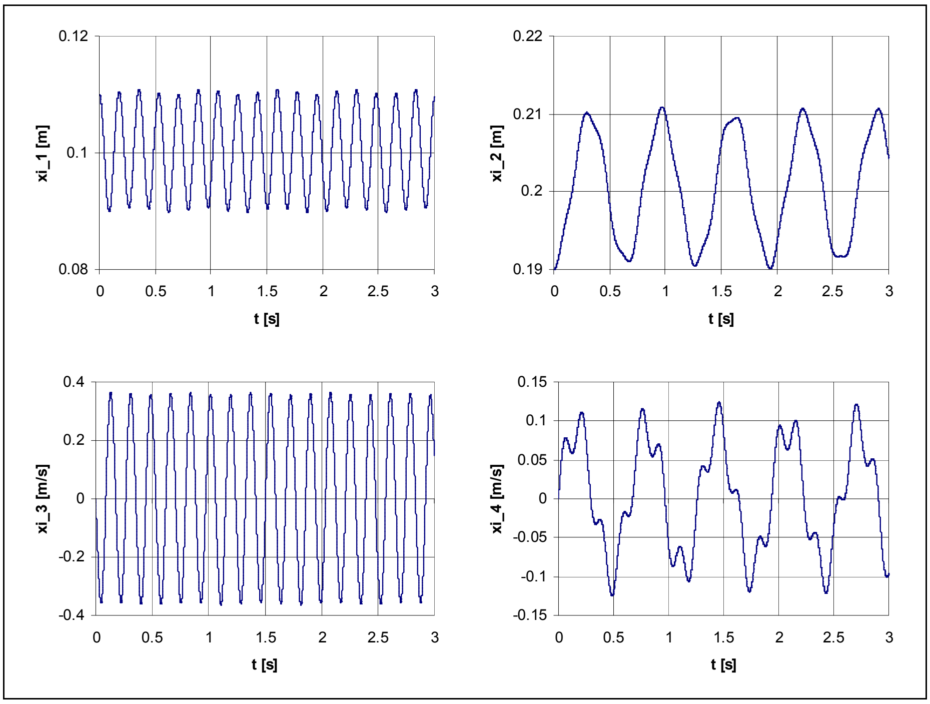

Figure 3 – The time history for the system with parameters (53) and the initial conditions (68).

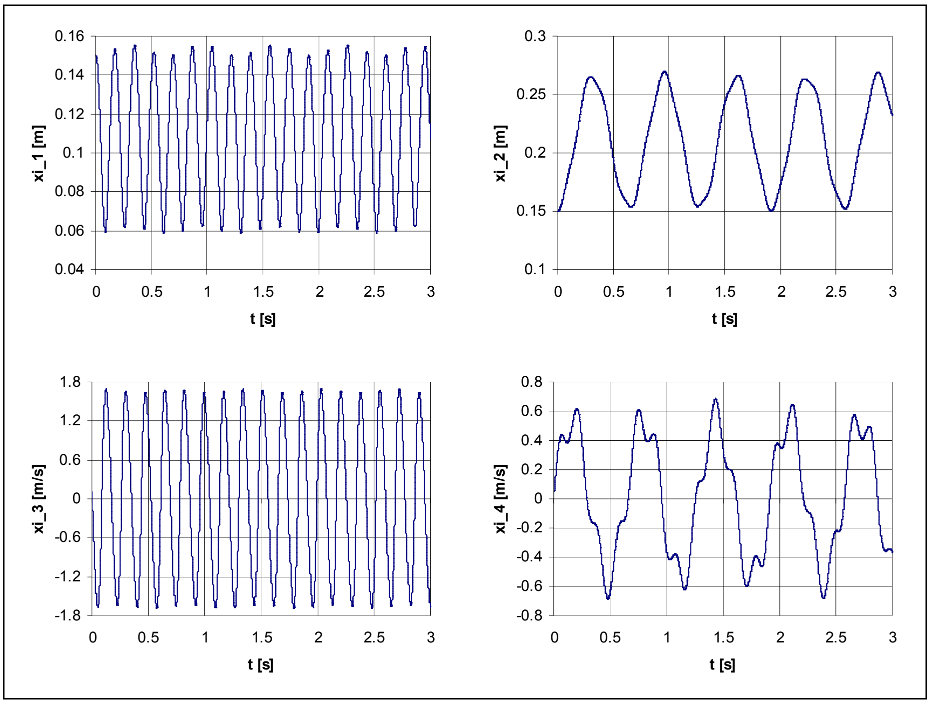

Figure 4 – The time history for the system with parameters (53) and the initial conditions (69).

In figure 3 are represented the diagrams $\xi_i(t)$, $i=\overline{1,4}$ for the parameters (53) the initial conditions being

$$\xi_{10} = 0.11\,\mathrm{m}\,,\ \xi_{20} = 0.19\,\mathrm{m}\,,\ \xi_{30} = 0\,,\ \xi_{40} = 0\,. \tag{68}$$

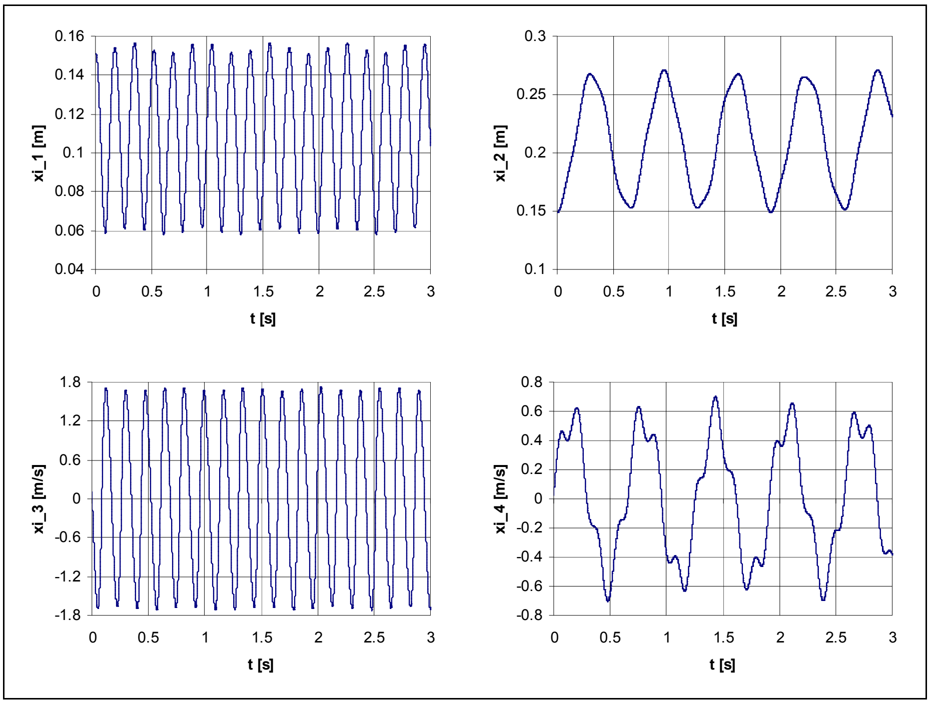

Figure 5 – The time history for the system with parameters (53) and the initial conditions (70).

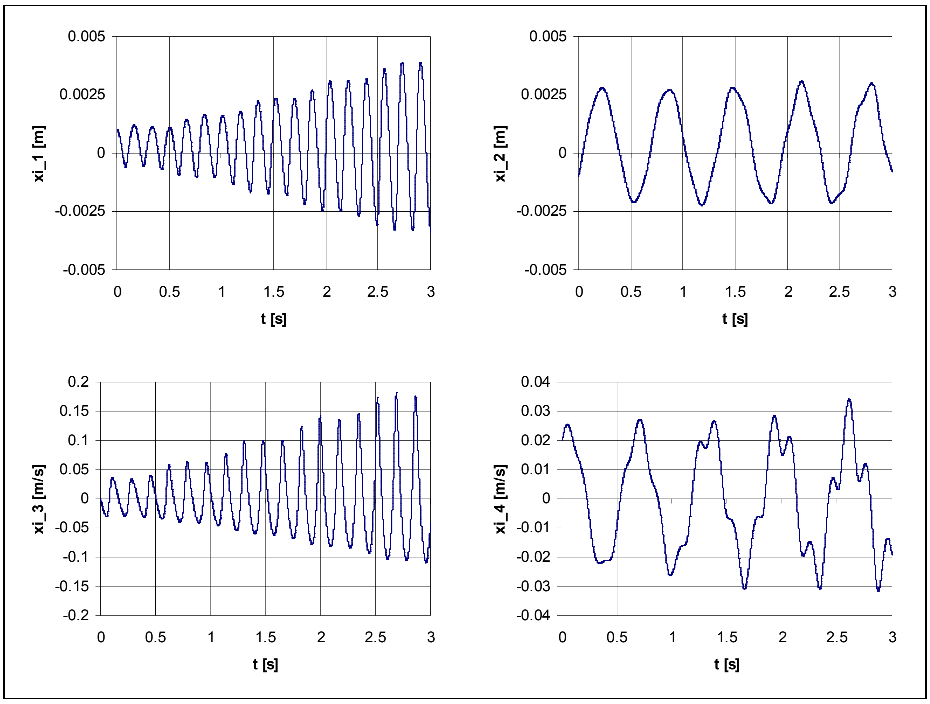

Figure 6 – The deviated motion between the systems captured in figures 4 and 5.

In figure 4 are represented the same diagrams for

$$\xi_{10} = 0.15\,\text{m}\,,\ \xi_{20} = 0.15\,\text{m}\,,\ \xi_{30} = 0.1\,\text{m/s}\,,\ \xi_{40} = 0\,, \tag{69}$$

in figure 5 for

$$\xi_{10} = 0.151\,\mathrm{m}\,,\ \xi_{20} = 0.149\,\mathrm{m}\,,\ \xi_{30} = 0.1\,\mathrm{m/s}\,,\ \xi_{40} = 0.02\,\mathrm{m/s}\,, \tag{70}$$

and in the figure 6 the deviated motion between the two cases from figures 5 and 6.

9. Conclusions

In this paper we presented a study concerning the influence of the non-linear neo-Hookean elements on the stability of the system machine-tool-foundation. We proved that both the equilibrium and the motion are simply stable and the neo-Hookean element increases the safety domain where the resonance doesn't appear.

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